INTERNATIONAL CENTRE FOR ECONOMIC RESEARCH

WORKING PAPER SERIES

François Gardes and Pierre Garrouste

HOUTHAKKER AND VILLE'S CONTRIBUTIONS TO DEMAND THEORY:

A NEW LOOK AT THE DEBATE ON INTEGRABILITY CONDITIONS

Working Paper No. 30 / 2004

Houthakker and Ville's contributions to demand theory: a new look at the debate on integrability conditions[1].

François Gardes[2]
Université de Paris I, Cersem, Crest (Lsm)
Pierre Garrouste[3]
Université Louis Lumière-Lyon 2, Atom

This version May, 2004

Abstract.

Jean Ville gave, independently of Houthakker, and prior to him, a general one page proof of the integrability of demand functions in a revealed preference scheme. It happens that this essential contribution has been largely ignored in the literature. The comparison between Ville and Houthakker's proofs makes room for discussing the assumptions necessary to encompass the discrete version of the acyclicity into a continuous version.

[1] We thank the International Center for Economic Research (Turin) and Cirano (Montreal) for their financial support. We acknowledge remarks by P. Crepel, P.Y. Jaffray, Alan Kirman, the participants to the Conference on The Theories of Value in Strasbourg and to the ENS-Cachan seminar.
[2] gardes@univ-paris1.fr
[3] galau@club-internet.fr

1. Introduction

In1946, Jean Ville published, in *Les Annales de l'Université de Lyon*, an article "Sur les conditions d'existence d'une ophélimité totale et d'un indice de prix". This very short article constitutes one of the more original French contributions to economics in the first part of the twentieth century and it is strange that in France economists forget it. In 1938, Samuelson published three articles[1] on the theory of utility that have initiated the Revealed Preferences Theory. In those contributions, Samuelson, thanks to the "Weak Axiom of Revealed Preferences" (WARP), establishes a theoretical result which obtains back the results of the ordinal utility theory without using any non-observational concepts. He uses only the notions of prices, quantities and income. It is a similar way of thinking that Ville proposes when he writes that «nous nous proposons dans le présent article d'étudier la formation des prix directement, à partir d'hypothèses revenant au fond a supposer l'existence de Φ [fonction d'ophélimité totale], mais plus directement exprimables en fonction des données fournies par l'observation. » (Ville, 1946, p.32)[2]. In fact, in the four assumptions he introduces, he uses prices, quantities and income, exactly in the same way Samuelson does in his revealed preference theory. However, while Samuelson, in his first articles (1938a, b, c), does not consider integrability as a pertinent problem, Ville demonstrates that the fact that there exists no close line along which the sum of the variations of the quantities time their prices keeps the same sign, is a necessary and sufficient condition for this sum to have an integrating factor. Samuelson changes his mind in 1950, and Houthakker (1950) evaluates the Samuelson's theory on the basis of the integrability problem.

Houthakker's proof is based on a discrete analysis which is transformed into a continuous one. He is considered as introducing the "Strong Axiom of Revealed Preferences" (the SARP). Ville is directly solving the integrability problem using continuous analysis. Indeed the shared idea is generally that the "Ville Axiom of Revealed Preferences" (VARP) can be transformed into the SARP by means of a small

[1] Samuelson (1938a, 1938b, 1938c).
[2] « We propose in the present article to study price-formation directly, starting from hypotheses which fundamentally, entail the existence of Φ, but which are more directly expressible as functions of data obtainable from observation. » (Ville, 1951-52, trad. Newman, p.123).

change. Hurwitz and Richter (1979) show however that those two axioms are not linked to the same properties of the Antonelli's matrix. The SARP implies the symmetry as well as the semi-definiteness of the Antonelli's matrix whereas the VARP lacks this last property. One of our aim is to explain this strange phenomenon and to analyze its consequences.

The remainder of this article is the following. Section two compares Samuelson (1938) and Ville (1946) and emphasizes on their common will to build up a theory of consumption only based on observable variables. In section three we present Houthakker's discussion of Samuelson revealed preference theory as well as his introduction of the SARP. In section four we discuss the Hurwitz and Richter (1979) arguments and present our own interpretation of the role of the Ville's axiom. Section five concludes.

2. Samuelson (1938) and Ville (1946) analysis

According to Samuelson, his knowledge of Ville's article is indirect and fuzzy. In Samuelson (1950) he writes: « Also there is a recent paper by (de) Ville which I know only from a brief review by K. Arrow in *Mathematical Reviews*, 1947 » (Samuelson, 1950, p. 358, note 8). Ville does not quote any economist (but Divisa page 38) and not at all Samuelson's works. An explanation of this lack can be found in the English translation of Ville's article. Indeed P. K. Newman writes : « M. Ville, a mathematician, first became interested in economics under rather peculiar circumstances. During his tenure of a professorship at Lyon(s), the shortage of academic staff forced him to lecture on 'financial mathematics', as well as on his regular subject-mechanics. He therefore, read several works in economics, being much influenced by Pareto's *Cours d'Economie Politique*. The theory of index numbers formed part of his course, and it was his attempt to put this difficult subject into some order that led to this article. Primarily, he sought definite answers to the question : 'When prices and incomes change, what can be said about the change in the individual's standard of living ?' » (Newman, 1951, p. 128, note)[3].

[3] In fact Ville was aware of economics. Pierre Crépel told us that in Lyon the economists and the mathematicians were close and were often discussing on economic problems.

We now compare Samuelson (1938a, 1938b, 1938c) and Ville (1946) on the basis of theirs objectives, their assumptions and their results and introduce Houthakker's analysis.

Partially common objectives

In 1938 Samuelson considers that the consumption theory based on the notion of utility progressively abandoned a whole part of its restrictive conditions: « (a) the assumption of linearity of marginal utilities; (b) the assumption of independence of utilities; (c) the assumption of the measurability of utility in a cardinal sense; and (d) even the assumption of an integrable field of preference elements. » (Samuelson, 1938a, p. 61). It is clear that (d) implies that Samuelson does not introduce the integrability assumption. For him the notion of utility is a psychological one, and its definition is empty and circular. « I propose, therefore that we start anew in direct attack upon the problem, dropping off the last vestiges of the utility analysis. » (Samuelson, 1938a, p. 62). The "last vestiges" referred to Hicks et Allen (1934) attempt to reconsider the value theory from the notion of marginal rate of substitution. In Samuelson (1938c), he writes in a very Popperian flavor: « it is the purpose here to demonstrate that the utility analysis in its ordinary form does not contain empirically meaningful implications by which it could be refuted. » (Samuelson, 1938c, p. 345).

In 1946, Jean Ville analyzes prices formation as well as the conditions of existence of an index of price level (something that is lacking in the Pareto's *Treatise*), and makes the assumption that there exists a utility function defined from observable data. In fact he thinks that the analysis of prices formation in economics lies on the assumption that there are some marginal utility functions that define a relation between the variation of the satisfaction, and the variation of the consumed quantities. However it is impossible to define a total utility function from those marginal utility functions because any monotonic function with its partial derivatives proportional to the marginal utility functions has the same properties relative to this function[4]. It is the reason why Ville

[4] This result is known since Fisher (1892) et Pareto (1896).

assumes the existence of a total utility function and tries to define both its existence conditions and its properties.

Different assumptions

In 1938 Samuelson's arguments start from the three following assumptions:

I. The demand functions are supposed known and are constrained by the individuals' budgets.

II. Those functions are homogenous of degree 0 in terms of the prices and income (this assumption is corresponding to the second of Ville's which considers that there is no money illusion).

III. « If an individual selects batch one over batch two, he does not select two over one. » (Samuelson, 1938a, p. 65).

Samuelson (1938b) shows that under some conditions, the third postulate is sufficient and implies the two firsts. It is the reason why the literature retains the "Samuelson's postulate". Samuelson is not only reluctant introducing the idea that the demand functions need to be integrable, but he considers that this condition is both unnecessary and unimportant. « Concerning the question of integrability I have little to say. I cannot see that it is really an important problem, particularly if we are willing to dispense with the utility concept and its vestigial remnants. (…) I should strongly deny, however, that for a rational and consistent individual integrability is implied, except possibly as a matter of circular definition. » (Samuelson, 1938a, p. 68).

Ville (1946, p. 32) introduces four «hypothèses relatives au comportement d'un acheteur de biens directement consommables »[5], which are almost similar too Samuelson's set of axioms:

I. The first one assumes that the quantities bought are only function of the set of the prices and the income of a buyer.

II. The second supposes that the buyer is not victim of some money illusion. Her behavior is sensitive only to the relative prices and her income (he writes in a

[5] « hypotheses relating to the behaviour of a buyer of directly consumable goods" (Newman, 1950-51, p.123)

note that this assumption is not necessary and shows that it is implied by the fourth one).

III. The third defines the consequences of a variation of the prices and income on the agent's standard of living: if an individual does not buy the same quantities after a variation of the prices and her income although she can do it then her standard of living has risen.

IV. The fourth assumption considers that if the prices and income describe a close cycle then the standard of living of a given individual cannot have increased on this cycle.

The last hypothesis is crucial because it permits to demonstrate that the sum of the variations of the prices multiplied by the corresponding quantities admits an integrating factor. The two first assumptions are common to Samuelson and Ville: first, the idea that the consumed quantities are perfectly defined by the prices of the goods and the income, second the assumption that the demand functions are homogenous of degree zero in terms of the prices and income are now classical ones. The third assumption contains the principle of Revealed Preferences and in the same time corresponds to the «Weak Axiom of Consumer's Behavior » (well known after as the Weak Axiom of Revealed Preference, i.e. WARP) of Samuelson (1950) because the modifications of the choices are linked with an increase of the "standard of living" (that is to say the satisfaction of the individual supposed independent of the prices). It is then impossible to take the reverse way, which is to prefer in another situation the second batch to the first, because the standard of living would have been decreasing. The fourth hypothesis contains a part of the « Strong Axiom » (SARP).

Indeed the WARP stipulates that: « if at the price and income situation A you could obtain the goods actually bought at a different point B and if you actually chose not to, then A is defined to be 'revealed better than' B. The basic postulate is that B is never to reveal to be *also* 'better than A'. » (Samuelson, 1950, p. 370) and the SARP : « if A reveals itself to be 'better than' B, and if B reveals itself to be 'better than' C, and if C reveals to be 'better than' D, etc…., then I extend the definition of 'revealed preference' and say that A can be defined to be 'revealed to be better than' Z, the last in

the chain. In such a case it is postulated that Z must never *also* be revealed to be better than A. » (Samuelson, 1950, p. 370-71).

The problem relies in the fact that this second definition does not correspond to what Samuelson writes in 1938 and is due to the existence of the Houthakker's article.

Samuelson's and Ville's results.

The results of Samuelson (1938a, 1938b, 1938c) are explicitly rooted in the debates concerning the consumer theory at this period. The first aim of Samuelson's theory, which is also in Ville (1946), is to show that it is possible to set up a consumer theory on the basis of the prices and income only. The second is that the WARP in Samuelson (1938b) is sufficient even if he writes that it is not permitting to find back all the Slutsky's results[6].

Samuelson defines a set of goods $(q_1,...q_n)$ with a set of corresponding prices $(p_1,...p_n)$ then another $(q_1',...q_n')$ of goods and a set of corresponding prices $(p_1',...p_n')$. Let $p.q = p_1q_1 +\ldots+ p_nq_n$ and $p.q' = p_1q_1' +\ldots p_nq'_n$. If the cost $p.q'$ is lower or equal to $p.q$, that means that the individual could have bought the quantities of the second set of goods at the prices corresponding to the first set and that he did not do it. This is expressed by:

$$p.q \geq p.q' \qquad (1)$$

In other words the first batch has been revealed to be preferred to the second.

[6] « for example one arrives at most of the properties of the Slutsky's matrix », Samuelson (1950, p. 370, note 1).

On this basis Samuelson introduces the anti-symmetry property of this relation showing that:

$$p'.q' \geq p.'q$$

implies that the second batch is preferred to the first which is in contradiction with $p.q \geq p.q'$. Then the reverse inequality must hold: $p'.q' < p'.q$ (2)

From (1) et (2), Samuelson derives its postulate III.

As written above Samuelson (1938b) shows that this postulate implies the two first assumptions. Thanks to this postulate he finds back the Georgescu-Roegen (1936) results, and he completely specifies the restrictive conditions on the demand functions.

Ville's results are in fact of two types. He demonstrates first the existence conditions of a total utility function (fonction d'ophélimité totale) defined in terms of the prices and income on the basis of hypotheses I to IV. Thus he "finds back" in a different way the Samuelson (1938a) principle of revealed preference of a set of consumed goods on a second one. He takes two arbitrarily close situations in terms of prices, quantities and income and shows that the cost in the second situation of the quantities consumed in the first one depends on the sign of the sum of the prices multiplied by the variations of the corresponding quantities. If this sum is positive (respectively negative) the individual's standard of living has increased (respectively decreased). This reasoning is exactly the same as Samuelson's, but Samuelson establishes that this sum is nil[7] and then bases his arguments on the anti-symmetry property of the relation between the batches of goods. Ville then demonstrates (see appendix A) that his hypothesis IV implies the integrability and by means of introducing the integrating factor of the sum of the prices multiplied by the variations of the corresponding quantities, he shows in only one page that there exists a total utility

[7] We will see below the consequences of this result.

function that respects the "classical" results in terms of maximization of utility, one of the interest of this proof being the introduction of symmetric expressions in terms of prices and quantities. He also shows that, depending on the existence of the total utility function, its lack of variation in time and the correct value of the integrating factor, it is then possible to calculate an index of the price level and of the standard of living.

3. Houthakker (1950) and the integrability question

Samuelson's results opened a new research program (Wong, 1978) which enlarged considerably the rather narrow point of view of his first articles. Considering that Samuelson's weak axiom is not sufficient in the general case, Houthakker introduces in 1950 the Strong Axiom of Revealed Preference to provide an axiomatic foundation for the existence of a utility function. « The main object of our investigation is to find a proposition which, *apart from continuity assumptions*, summarises the entire theory of the standard case of consumer's behaviour (…) Samuelson's hypothesis does not satisfy this criterion, being only a necessary condition and not a sufficient one, for although it can be derived from utility considerations it does not entail integrability, which is an essential property of utility functions. » (Houthakker, 1950, p.161). Houthakker uses the two first axioms proposed by Samuelson (1938a) and Ville (1946), but introduces first his strong axiom, so that these axioms seem to be embedded in his strong axiom, which seems to be equivalent to the third and fourth hypotheses made by Ville : « *if* $X^0, X^1, X^2 \ldots X^T$ *is a sequence of batches of goods such as each batch is bought at prices* $P^0, P^1, P^2 \ldots P^T$ *respectively, and if at least two of these batches are different, and if the cost of each batch* X^t *at price* P^{t-1} *is not greater than the cost* $P^{t-1} X^{t-1}$ *of the preceding batch in the sequence* X^{t-1} *at the same prices, then the cost* $P^T X^T$ *of the last batch* X^T *at prices* P^T *is less than the cost* $P^T X^0$ *of the first batch* X^0 *at the same prices*. » (Houthakker, 1950, p.163).

Samuelson soon recognizes in 1950 the importance of supposing the impossibility of closed paths: « I soon realised that this could carry us almost all the way along the path of providing new foundations for utility theory. But not quite all the way. The problem of integrability, it soon became obvious, could not yield to this weak

axiom alone »… « But no proof was forthcoming for all these years, until Mr. Houthakker's paper arrived in the daily mail. » (Samuelson, 1950 b, p. 371). It is interesting to note here that Corbett and Newman (1952-53) do not refer to Ville's article although the second translates Ville's article (1951-52). In fact criticizing Houthakker's proof, they use Georgescu-Roegen's results (1936) to stress the necessity of transitivity in order to discuss the existence of indifference curves and to imply integrability[8].

Contrary to Samuelson's reasoning, Ville's proof does not refer to preferences or indifference, so that it is not concerned by Little's « semantic » critics. Ville's proof is completely apart from the debates between economists (he does not for instance suppose that the sum of prices weighted by the changes in quantities is null, which is implied by Samuelson's third axiom, allowing to define isoquants). Ville is not obliged, as does Samuelson when refusing the utility theory, to compare his results to Georgescu-Roegen theory, and his proof is much more elegant from the mathematical point of view while it finally contains implicitly the theory of the ordinal utility without referring explicitly to it[9].

Despite these correspondences, Ville's axiom is not equivalent to Houthakker's strong one: for differentiable demand functions, it is a consequence of the strong axiom. However, the strong axiom implies more than the integrability of demand functions: it does imply the semi-definiteness of the Antonelli's matrix, which is equivalent to the

---

[8] In fact, Georgescu-Roegen did not really showed that transitivity implies integrability, but just that integrability is not sufficient to imply the existence of indifference curves : « the preceding analysis shows clearly that without the transitivity postulate the integral varieties and the indifference varieties are two distinct things. If a point of saturation exists, the indifference varieties will be always concave closed surfaces (postulate A) while the integral ones need not necessarily be so. The much-discussed paradox of the non-integrable case is due to the confusion of these two concepts. » (Georgescu-Roegen, 1936, p. 508). The same confusion appears in Samuelson's article (1950), where the transitivity and integrability are not clearly distinguished, perhaps because they are both imbedded in Houthakker's strong axiom. On this point, Ville's proof is more consistent, since he does not assume directly transitivity to obtain integrability.
Note also that the continuity assumptions made by Houthakker are very clumsy. They deserved a special treatment by Dagum (1973), and are in fact at the core of the difference between Houthakker and Ville proofs.

[9] Hurwicz-Richter (1979) note that this may lead to some misunderstanding of his results : « Because he uses preference terminology rather than revealed preference terminology, he might at first glance appear to employ circular reasoning, assuming a preference exists in order to prove a preference exists » (pp. 607-8).

weak axiom and thus is not sufficient to obtain Ville's conditions. As shown by Richter-Hurwicz (1979), the acyclicity contained by Ville's conditions is a necessary and sufficient condition for the symmetry of the Antonelli's matrix, though it is not sufficient for its the semi-definiteness, and thus it does not imply the weak axiom, as will be discussed in the next section.

4. VARP, SARP and WARP reconsidered

In an article published in 1979, Hurwicz[10] and Richter give a slightly more general version of Ville's proof qualified as « a most important contribution to the axiomatic of the consumer theory » (p. 603). Their theorem 1 (p. 609) show that if no continuous cycle exists in the neibourghood of some point, the Antonelli's matrix is symmetric over another neibourghood of the same point, thus implying the integrability of the corresponding demand functions. They show also the reverse: the symmetry of the Antonelli's matrix around some point implies the absence of continuous cycles in the neibourghood of this point. The crucial consequence of their analysis is that Houthakker's strong axiom, because it encompasses the weak axiom, also implies the semi-definiteness of the Antonelli's matrix: the strong axiom is too strong because it is not equivalent to the sole symmetry of the Antonelli's matrix which is sufficient for obtaining integrability.

Ville's axiom corresponds to writing the strong axiom in continuous terms : in SARP, every path composed by bundles q^1, …, q^k and the vector of prices and income (p^k, y^k) cannot contain any cycle of revealed preferences. This means that the following inequalities between consecutive situations :

$$P(\tau_2).[x(\tau_2) - x(\tau_1)] > 0 \qquad (1a)$$

cannot be verified all along the cycle, whatever it is.

[10] See also Hurwicz (1971) and Hurwicz and Uzawa (1971).

The continuous version of this assumption says that, along a continuous cycle, the inequality:

$$P(\tau).dx/dt(\tau)>0 \qquad (1b)$$

is not always verified.

The simple fact to consider the continuous version of axiom (1b) makes the semi-definiteness of the Antonelli's matrix vanished. Ville's axiom thus appears as the minimal condition to imply integrability, while Houthakker's proof in not satisfying, since it assumes a stronger hypothesis. There lies the true difference between the two theories, and not, as often mentioned, because inequalities are written in their strict or large version.

One may note that the discrete version (1a) would be obtained by integrating the differential condition (1b) over finite intervals. On the contrary, what prove Hurwicz-Richter equivalences is that *this integration is only possible under some regularity condition of the corresponding demand functions,* a condition which is given by the semi-definiteness of the Antonelli's matrix. Therefore, the discrete assumptions cannot be deduced from the continuous which is the only necessary assumption for integrability.

The first difficulty lies in the algebraic or differential form of the axioms. For the weak axiom, which compare only two situations, no difference exists between these two versions, but the difference is important for the strong axioms: Ville's axiom is explicitly differential, relating to continuous preference cycles, while Houthakker's one compares a finite number of situations which are pushed to the limit by a rather intricate reasoning. By the way, the role of the continuity hypothesis is not clear in Houthakker's analysis, as are also the equivalences between the axioms and the properties of demand functions and Slutsky's or Antonelli's matrices.

Second, the weak axiom (which is just equivalent to the positive semi-definiteness of the Slutsky's matrix, see for instance John, 1995) corresponds to a hypothesis of *consistency* (i.e. anti-symmetry) between two situations x et x + dx, that is to the *stability of preferences* from a dynamic point of view. Ville's axiom implies the symmetry of the Antonelli's matrix which makes it possible to integrate demand functions into an utility function, and thus the possibility of *comparing* all situations (for the same reason, Arrow , 1959, 1974, must add an hypothesis of comparability to his weak axiom to prove the existence of a utility, see also Campbell, 1994).

Finally, SARP compares a much greater number of situations than WARP, which justifies its name. Each of these two axioms corresponds to a different property of the Slutsky's and Antonelli's matrices, the empirical content of which is still largely unknown. A rapid combinatory analysis of the situations compared by these axioms tends to show that the strong axiom compares a number of situations in proportion (n-2), with n the number of bundles, to the number of situations compared by the weak axiom. However, an empirical analysis of rationality conditions on a Polish panel (Diaye, Gardes, Starzec, 2001) tends to show that the violations of WARP are as numerous as those of SARP and GARP (the Generalized Axiom defined by Afriat), which seems to indicate that their empirical contents are not ordered as would be normally assumed.

5. Conclusion

Why this silence about Ville's contribution, despite Arrow's discussion as soon as 1947 and its early translation (1951) by Newman? Its mathematical content is easily understandable, even by economists, and so pure that, contrary to Houthakker's one, once understood, it is completely mastered. Moreover, it is uneasy to find in it any logical contradiction, while the convergence arguments of Houthakker's article are highly contestable since Houthakker uses some continuity assumptions in his original

proof which are not sufficiently defined (see specially pp. 165-7 and the implicit assumption of uniform continuity. See also Dagum, 1973, p. 411[11]).

The splendid simplicity of Ville's proof may be though by economists as canceling all depth to the economic view of the integrability problem, which yielded so many controversies and is solved so rapidly by means of an elementary application of thermodynamics principles using a classic property of Pfaff's forms. Also Ville does not popularize his theorem among English speaking academic circles. He is also profoundly engaged in research on game theory and probability, giving the first geometric proof of the Minimax theorem of Von Neuman and introducing, as early as in 1930, the concept of martingale, but he is never fully recognized, and never tries to, as a great innovator even by the French mathematicians in these field, despite his marvelous contributions. We must stress finally :

(i) that Jean Ville gives independently of Samuelson and Houthakker, in his third hypothesis, section 1, a definition of revealed preference theory,
(ii) Defines in the same time the weak axiom,
(iii) And gives a one page proof of integrability conditions which is both the first and the best for its axiomatic completeness and economicity.

The comparison between Ville's and Houthakker's proofs makes room for discussing the assumptions necessary to encompass the discrete version of the acyclicity into a continuous version, making assumptions about the properties of the choice sets (since it appears that these axioms postulate the comparability between different bundles of goods and the path independence of preference relations).

[11] Dagum discusses the hypothesis that « the two sequences of income functions corresponding to Houthakker's upper and lower sequences of offer curves possess subsequences which converge to a function that satisfies the « right » differential equation ».

References.

Allen, R. G. D. and Hicks, J. R., 1934, A reconsideration of the Theory of Value, part I, *Economica*: 52-76 ; part II, *Economica*: 196-219.

Arrow, K., 1974, *Choix collectif et préférences individuelles*, Paris : Calmann-Levy.

Campbell, D. E., 1994, Arrow's Choice Axiom, *Economic Letters* 44: 381-84.

Chipman, J. S., Hurwicz, L., Richter, M. K., Sonnenshein, H. F., 1971, *Preferences, Utility and Demand*, New York : Harcourt Brace Janovich

Corlett, W. J. and Newman, P. K., 1952-53, A Note on the Revealed Preference and the Transitivity Condition, *Review of Economic Studies*, 20: 156-58.

Dagum, B. P., 1973, Revealed Preferences- a Proof of Houthakker's Theorem, *Econometrica*, vol. 41, n° 3, May : 411-423.

Diaye, M., Gardes, F., Starzec,C., 2001, The World According to GARP, Cahiers de recherche du CREST.

Fisher, I., 1892, *Mathematical Investigations in the Theory of Value and Prices*, Transactions of Connecticut Academy of Arts and Sciences, 9 : 1-124.

Georgescu-Roegen, N., 1936, The Pure Theory of Consumer's Behavior, *Quarterly Journal of Economics*, 50 : 545-93.

Georgescu-Roegen, N., 1950, The Theory of Choice and The Constancy of Economic Law, *Quarterly Journal of Economics*, 64 : 125-38.

Georgescu-Roegen, N., 1954, Choice, Expectations and Measurability, *Quarterly Journal of Economics*, 48 : 503-34.

Houthakker, H.S., 1950, Revealed Preference and the Utility Function, *Economica*, 17 : 159-74.

Houthakker, H. S., 1961, The Present State of Consumption Theory : A Survey Article, *Econometrica*, 29 : 704-40

Hurwicz, L., 1971, On the problem of Integrability of Demand Functions, in Chipman, J. S., Hurwicz, L., Richter, M. K., Sonnenshein, H. F. (eds) (1971) *Preferences, Utility and Demand*, New York: Harcourt Brace Janovich

Hurwicz, L. ; Richter, M. K., 1979, Ville Axioms and Consumer Theory, *Econometrica*, 47, 3 : 603-619.

Hurwicz, L. and Uzawa, H., 1971, On the problem of Integrability of Demand Functions, in Chipman, J. S., Hurwicz, L., Richter, M. K., Sonnenshein, H. F. (eds) *Preferences, Utility and Demand*, New York: Harcourt Brace Janovich

John, R, 1995, The Weak Axiom of Revealed Preference and Homogeneity of Demand Functions, *Economic Letters*, 47 : 11-16.

Little, I. M. D., 1949, A Reformulation of the Theory of Consumer's Theory, *Oxford Economic Papers*, 1 : 90-99

Little, I. M. D., 1950, *A Critique of Welfare Theory*, Oxford : Oxford University Press.

Pareto, V., 1896, *Cours d'Economie Politique*, Lausanne : Rouge.

Samuelson, P.A., 1938a, A Note on the Pure Theory of Consumer's Behaviour, *Economica*, 5 : 61-71.

Samuelson, P. A., 1938b, A Note on the Pure Theory of Consumer's Behaviour : An Addendum, *Economica*, 5 : 353-354.

Samuelson, P. A. ,1938c, The Empirical Implications of Utility Analysis, *Econometrica* : 344-56.

Samuelson, P. A., 1948, Consumption Theory in Terms of Revealed Preference, *Economica* : 243-53.

Samuelson, P.A., 1950, The Problem of Integrability in Utility Theory, *Economica*, 17 : 355-85.

Ville, J. , 1946, Sur les conditions d'existence d'une ophélimité totale et d'un indice de prix, *Annales de l'Université de Lyon*, 9 : 32-9, traduit par Newman, P.K. (1951-2) : The Existence Conditions of a Total Utility Function, *Review of Economic Studies*, 19 : 123-8.

Wong, S. ,1978, *The Foundations of Paul Samuelson's Revealed Preference Theory*, London : Routledge & Kegan Paul.

Annex: Ville's demonstration of the integrability conditions.

1. Ville considers the demand functions as depending of the prices and income of the consumer:

$q_i = f_i (p_1,\ldots, p_n,s)$, supposed homogenous of degree 0 (and continuous).

2. The cost of the quantity vector q at prices $p'=p+dp$, is:

$p'.q = (p+dp).q = p.q+(p.dq+q.dp+dp.dq)–p.dq–dp.dq$

$= s+ds–p.dq–dp.dq$

$\approx s+ds–p.dq$ at the first order, then : $p'.q < s+ds \Leftrightarrow p.dq > 0$ (1) (it is important to notice that Ville reasons in terms of strict inequalities. That permits to pose the problem at the first order using the continuity of the demand functions).

The first inequality : $p'.q < s+ds$, implies that the vector of quantities of chosen goods at prices $p+dp$ and for an income $s+ds$ is preferred to the vector of initial chosen quantities at prices and for a vector de quantities chosen for the initial prices and income because it corresponds to a higher cost. It is the principle of the revealed preference: the consumer could continue to choose vector q at new prices p', because the cost would have been lower than the cost of $q+dq$. If she chooses $q+dq$, it shows that she strongly prefers this consumption, this new vector being more expensive than the initial one. We obtain with equivalence (1):

$p.dq > 0 \Leftrightarrow q(p+dp,s+ds) > q(p,s)$.

Therefore, the sign of $p.dq$ indicates the preference between $q(p,s)$ and $q(p+dp,s+ds)$.

3. The fourth Ville's assumption stipulating that there is no cycle along a closed contour, which is equivalent that $p.dq$ cannot constantly increase if the standard of living does not continuously increase, implies that $p.dq$ has an integrated factor. The demonstration is the following:

(i) If $p.dq$ has not an integrated factor it is possible to put it under one of two Pfaff's forms[12] :

[12] $V_i(x_1, x_2, \ldots,x_n)$ being a n vector field (i=1, 2, …, n), the form $V_1.dx_1+ V_2.dx_2+\ldots+ V_n.dx_n$ is called a Pfaff's form (Pfaff forms have been extensively studied by Elie Cartan, a French mathematician probably well known by Ville). When the V_i are the coordinates of the gradient of a differentiable function on a open set of ∇^n, then the Pfaff's form is the total differential of this function.

$$d\varpi = dz - y.dx = dz - \sum_i y_i dx_i,$$

or:

$$d\varpi = -y.dx$$

with a number h of variables y_i et x_i less or equal to the number n of goods. The new variables are in a number, $2h+1$ or $2h$, less or equal to the number n of variables q_i. Ville notes that the first form is in general the case for an odd number of goods, the second for an even number.

(ii) He defines the closed outline as the following:

$$x_i = \cos \varphi$$

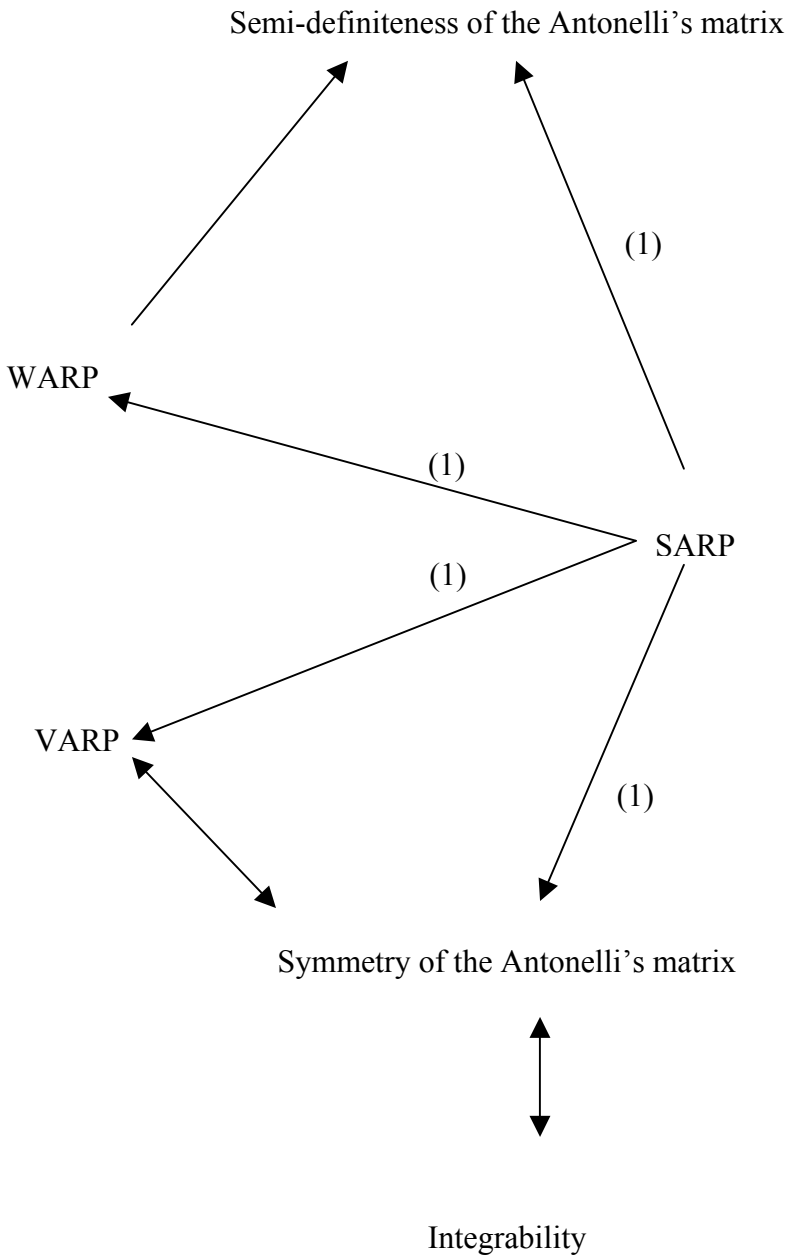
$$y_i = \sin \varphi$$

$$z = -\cos \varphi + 1/3 \cos^3 \varphi.$$

The vector (x, y, z) describes the outline when φ goes from 0 to 2π, and then:

$d\varpi = (h + \sin\varphi) \sin^2\varphi \, d\varphi$ for the first form (with $h>1 \Rightarrow h + \sin\varphi > 0$), or : $h \sin^2 \varphi.d\varphi$ for the second, both keep a constant sign all along the closed outline, which contradicts the fourth assumption. The writing under the Pfaff's form being impossible, $p.dq$ has an integrated factor.

Annex: a synthetic figure.



(1) Under continuity conditions